MATHEMATICS SL INVESTIGATION

Methods of approximating sin(x) as an algebraic function

Introduction

The function $sin(x)$ is an elementary trigonometric relation often defined in terms of the unit circle as the vertical distance between the x-axis and the point on the unit circle that meets a line subtended by angle x (in radians). It exhibits the property of periodicity in that its values and the shape of its graph are repeated after an interval of 2π. This makes the function an essential tool in developing mathematical models and generalised formulas for periodic phenomena such as temperature cycles or tidal wave behaviour. However, deriving values for $sin(x)$ is surprisingly difficult without the use of a calculator or other computerised machinery. Unlike algebraic functions, which can be expressed in terms of a finite combination[1] of algebraic operations, $sin(x)$ is a **transcendental function;** it can only be defined in terms of other trigonometric expressions, or an infinite series of polynomials. For centuries, solving for sines often meant tediously examining an elaborate trigonometric table and finding the corresponding value of $sin(x)$ for each required instance of x. This is an impractical process and having to "search" for the values of a function seems counter-intuitive to the search for generalisation and broad applicability in mathematics.

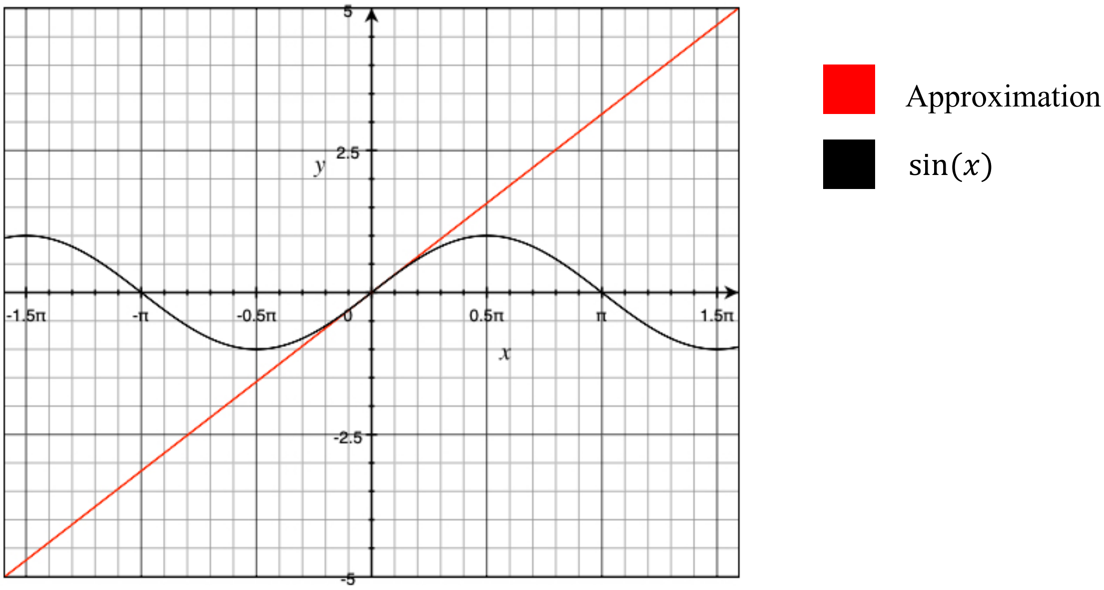
In my IB physics class, we were taught to replace the function $sin(x)$ with simply the angle x itself for any "small" values of x in radians. It allowed us to derive a much simpler formula for wave diffraction without compromising the theoretical accuracy of the formula. I became increasingly curious about what "small" truly meant, and in my quest to understand the limits of this approximation, I realised that $sin(x)$ — or any other function — could actually be algebraically approximated in a number of ways. This investigation will explore the mathematical accuracy, rationale, and practical limitations of a few key methods of approximating the function $sin(x)$. The success of each approximation depends

[1] "Transcendental Function." *Encyclopædia Britannica*, Encyclopædia Britannica, Inc., 15 Apr. 2011, www.britannica.com/science/transcendental-function. Accessed 27 Jan. 2020.

on its ability to map $sin(x)$ to a simpler algebraic form that would enable direct calculation and allow us to make less obtrusive models without compromising significant accuracy.

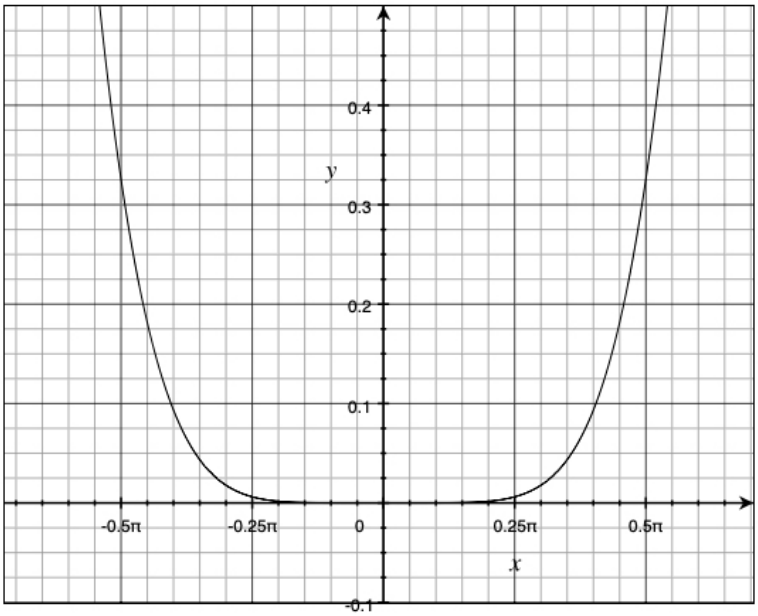
<u>Substituting x for sin(x): "The Small-Angle Approximation"</u>

The aforementioned method of approximating $sin(x)$ to x for "small angles" is a common practice in mechanical physics. The accepted rationale behind this substitution is that the graph of $sin(x)$ roughly coincides with the graph of $f(x) = x$ for angles very close to zero. Plotting the graph of both functions in the domain -2π to 2π shows that they are *seemingly* comparable around the origin.



Graph 1: The small-angle approximation and sin(x)

However, visual similarity at the low resolution of this document is hardly rigorous mathematical confirmation. We can plot an error squared function for the approximation by taking the square of the difference between the two functions: $(sin(x) - x)^2$.



Graph 2: The squared error in the linear approximation (sin(x) - x) [2]

We observe that the squared error is less than 0.00001 for angles between $-\frac{\pi}{12}$ and $\frac{\pi}{12}$, which would imply that this linear approximation is accurate up to two decimal places in this domain. It is important to note that the error reduces consistently as x approaches zero, so the smaller the angle, the better the approximation. In the motion of a simple pendulum, where maximum angular displacement θ is most often below $\frac{\pi}{12}$, using this approximation can give us a simpler equation for calculating the acceleration of the pendulum bob (where L is the length of the pendulum cord).

The theoretical formula is:

$$a = -gsin(\frac{\theta}{L})$$

Using the small-angle approximation, we eliminate the sine function entirely, leaving us with:

$$a = -g(\frac{\theta}{L})$$

Although our function loses its mathematical accuracy to a certain extent, it is now possible to calculate a number of physical quantities with simple algebraic operations, including the acceleration of the bob and the force exerted by it in its direction of motion. In cases such as this, where the domain of x is known to be restricted to values near zero ($x \rightarrow$

0), equating $sin(x)$ to x can be practically useful. The definition of "small angle" — the range in which this approximation can be used — depends on the level of accuracy that is appropriate or necessary for a given situation but taking smaller angles will reduce the error in the approximation increasingly as x approaches zero.

Simple Quadratic Approximation

The key issue with using linear approximations for trigonometric functions is that they cannot mimic the curvature of the graphs of such functions, and thus an approximation such as $sin(x) = x$ becomes extremely inaccurate as $sin(x)$ approaches its first maximum at $\frac{\pi}{2}$. We can obtain a simpler function that approximates the parabolic shape of $sin(x)$ between 0 and π by using quadratics instead[2].

We know that any quadratic function can be expressed in the form $ax^2 + bx + c$, where a, b and c are the coefficients of the expression and $a \neq 0$. The graph of $sin(x)$ in the domain 0 to π resembles a downward facing parabola, with roots 0 and π, and a maximum at $\frac{\pi}{2}$. Using this information, we obtain 3 simultaneous equations that we can solve for a, b and c:

$$0 = a(0) + b(0) + c$$
$$1 = a(\frac{\pi}{2})^2 + b(\frac{\pi}{2}) + c$$
$$0 = a(\pi)^2 + b(\pi) + c$$

Which gives us the values: $a = \frac{-4}{\pi^2}, b = \frac{4}{\pi}, c = 0$. Plugging these into the quadratic formula, the equation becomes:

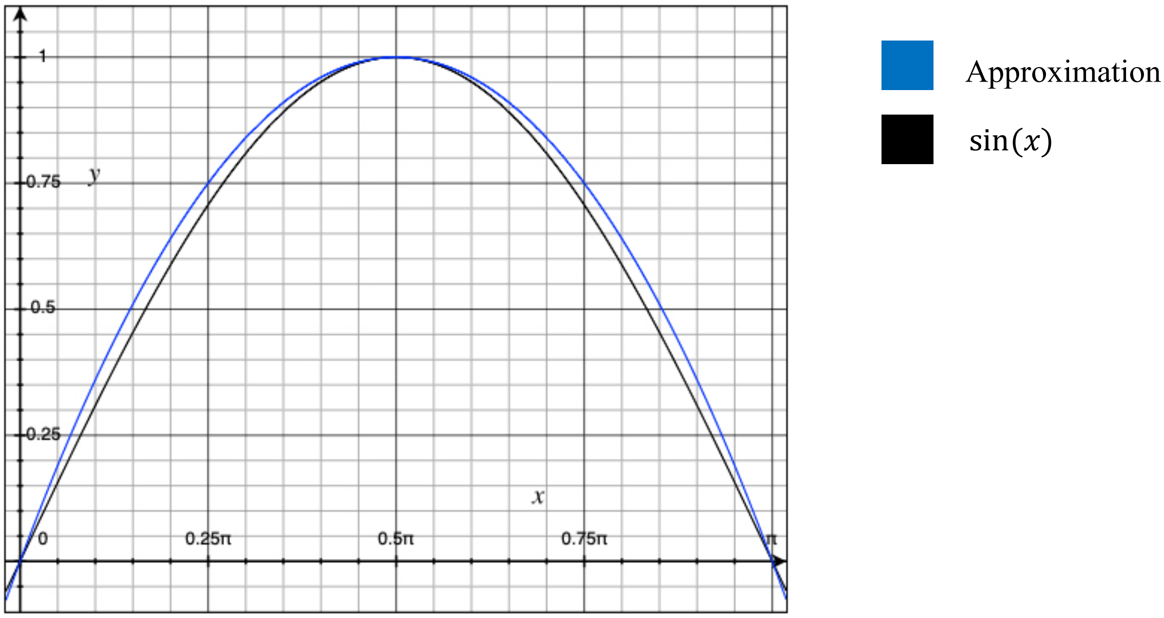
$$y = \frac{-4x^2}{\pi^2} + \frac{4x}{\pi} + 0$$

Using π^2 as a common denominator, we get:

[2] Berry, Nick. "Approximating the Sine Function." *Data Genetics*, 1 July 2019, datagenetics.com/blog/july12019/index.html. Accessed 3 Dec. 2019.

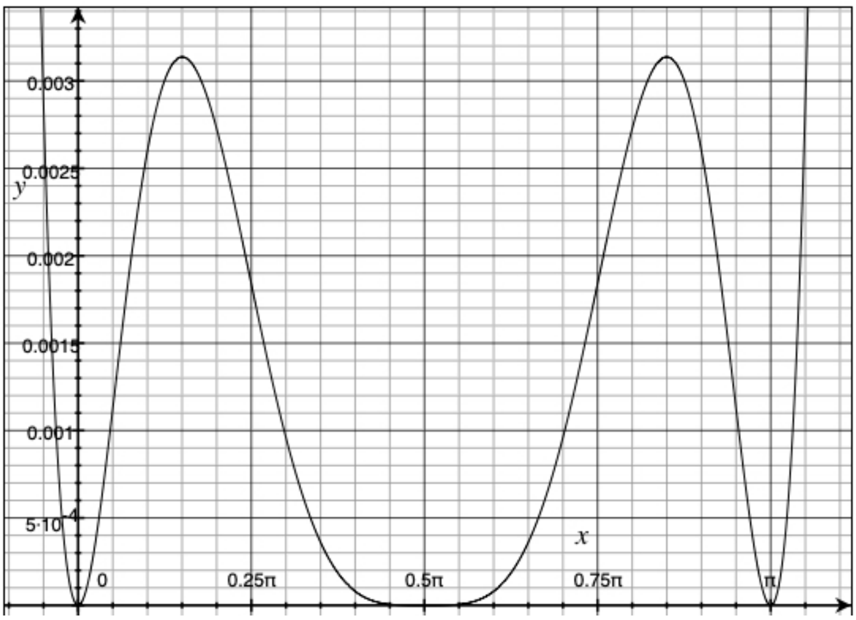
$$\rightarrow y = \frac{4x(1 - \frac{x}{\pi})}{\pi}$$

$$\rightarrow y = \frac{4x(\pi - x)}{\pi^2}$$



Graph 3: The simple quadratic approximation plotted against sin(x)

Plotting the two functions together in Graph 3, we can see that the approximated parabola does roughly resemble the shape of the sine graph between 0 and π, with the two graphs becoming closer to each other at the 3 points we used to formulate the approximation — (0,0), $(\frac{\pi}{2}, 1)$ and $(\pi, 0)$.



Graph 4: Squared-error in the quadratic approximation

The peak value of the error graph in Graph 4 is 0.003137 between 0 and π, showing that this approximation is fairly accurate up to 1 decimal place in the chosen domain. The fixed points we used to derive the quadratic function are still present in this approximation, which means that it will be even more accurate for values surrounding 0, $\frac{\pi}{2}$ and π. The fundamental issue with using such an approximation is that it would become highly inaccurate beyond this domain due to the nature of quadratic functions — while a pure sine function has an endless (infinite) number of upward and downward curves, a quadratic function will only formulate one parabola, and will move away from the x-axis continuously for any values of x beyond that. This method would be appropriate and efficient for approximating $sin(x)$ in certain cases where the range of the function required is relatively small (up to π) as quadratic graphs only have two roots, but they can roughly mimic the shape of the sine curve in any one direction. In order to form approximations with more roots and thus cover a larger domain, we must use polynomials of a higher degree.

Taylor Polynomials and Higher Degree Approximations

The Taylor series, also known as the Maclaurin series when it is centred at zero, is a method of representing any function as an infinite sum of algebraic terms using various derivatives of the original function[3]. Theoretically, the infinitely long Taylor series of a given function will be identical to the original function, but if we take a limited number of terms in the series (or "truncate" it), we can obtain useful approximations. The formula for calculating Taylor polynomials is as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

[3] "Taylor Series Approximation." *Brilliant Math & Science Wiki*, brilliant.org/wiki/taylor-series-approximation/. Accessed 12 Feb. 2020.

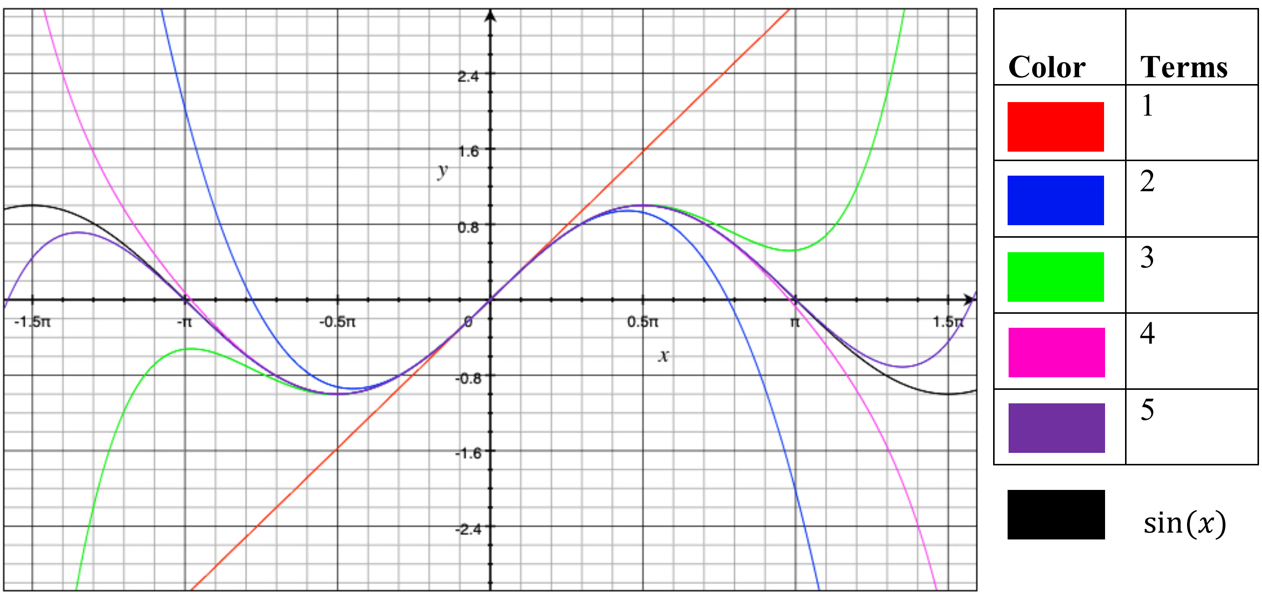
Unlike quadratic functions, which can only produce a single parabolic curve, using the Taylor series can produce higher degree polynomial approximations. Let us denote the Taylor polynomial for $sin(x)$ where $a = 0$:

$$sin(0) + \frac{cos(0)}{1!}(x-0) - \frac{sin(0)}{2!}(x-0)^2 - \frac{cos(0)}{3!}(x-0)^3 + ..$$

As any term involving $sin(0)$ will have a value of zero, the series becomes:

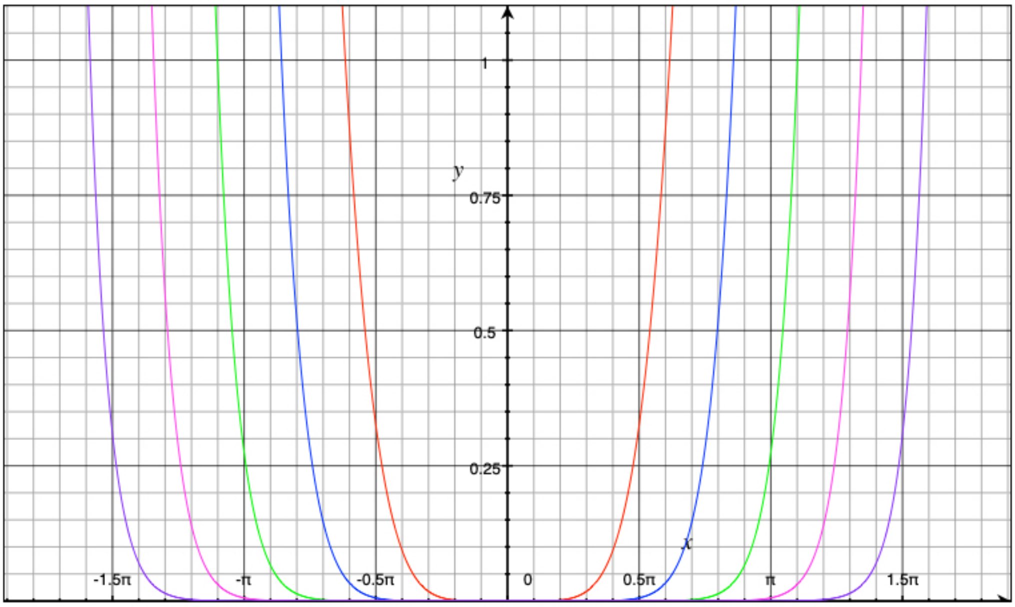
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We can see that the first term in this series is in fact x, the small-angle approximation for $sin(x)$. Plotting the first 5 terms of this function over the graph of $sin(x)$ allows us to observe an intriguing pattern:



Graph 5: The first five Taylor polynomials for sin(x) centered at 0

Each additional term adds two new roots to the approximation and makes it closer in shape to the pure sine function. The higher degree approximations also appear to have a higher range of Y-values in which they are accurate. Seemingly, taking a higher degree approximation would allow us to gain a more accurate and versatile approximation. Let us test this by plotting the absolute error function in each of these polynomial approximations in Graph 6:



Graph 6: The squared error in the Taylor polynomials

Although all of our Taylor approximations have an absolute error of zero for $x = 0$ (as this is the value on which our series is centred), the polynomials of a higher degree are observed not only to have larger domains where they are relatively accurate, but also a lower absolute error for *all* values of x. That is to say, including more terms of the Taylor series in our approximation will give us a value that is closer to the pure sine function.

Bhaskara I's Approximation

Bhaskara I was a seventh-century mathematician[4] who derived one of the earliest yet most elegant approximations of $sin(x)$. It is a rational function composed of a ratio of two quadratic functions, and it is known to provide accurate approximations of trigonometric values despite its simple form. Bhaskara I's formula can be written as follows:

$$sin(x) \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$$

[4] O' Connor, J. J., and E. F. Robertson. "Bhaskara I." *Bhaskara I (about 600 - about 680)*, mathshistory.st-andrews.ac.uk/Biographies/Bhaskara_I.html. Accessed 3 Dec. 2019.

Bhaskara I never provided a detailed rationale or proof for his formula, but it is indeed possible to derive using algebraic methods. This formula can be seen to bear resemblance to the previously derived quadratic approximation — both functions contain the quadratic relation $4x(\pi - x)$ in their numerator, which means that Bhaskara I's approximation can be expressed as a scaled expression[5] of our original quadratic approximation. This essentially involves adding a "scale factor" to our original formula that increases its accuracy by using another set of known values for $sin(x)$.

Since the scaling function is a quadratic, we start with:

$$\frac{\frac{4x(\pi - x)}{\pi^2}}{f(x)} = \frac{\frac{4x(\pi - x)}{\pi^2}}{ax^2 + bx + c}$$

As the value of our quadratic approximation was exactly equal to the actual value of $sin(x)$ at $\frac{\pi}{2}$, we know that the scale function will be equal to 1 at this point. To find a, b and c, we can take two other points on the graph of $sin(x)$ that we know to be inexact in our approximation: $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. By scaling our original quadratic approximation, we can ensure that the new formula returns the exact values for $sin(x)$ at these two points, which means that it will be more accurate than the previous one and the shape of its graph will coincide more closely with $sin(x)$.

Using the information $sin(\frac{\pi}{6}) = sin(\frac{5\pi}{6}) = \frac{1}{2}$, we can derive our scaling function. The function $\frac{4x(\pi-x)}{\pi^2}$ returns a value of $\frac{5}{9}$ for both of these values of x, which means that our scale factor must be $\frac{10}{9}$ for the approximation to equal $\frac{1}{2}$. This gives us 3 simultaneous equations for $f(x)$:

$$\frac{10}{9} = a(\frac{5\pi}{6})^2 + b(\frac{5\pi}{6}) + c$$

$$\frac{10}{9} = a(\frac{\pi}{6})^2 + b(\frac{\pi}{6}) + c$$

[5] Berry, Nick. "Approximating the Sine Function." *Data Genetics*, 1 July 2019, datagenetics.com/blog/july12019/index.html. Accessed 3 Dec. 2019.

$$1 = a(\frac{\pi}{2})^2 + b(\frac{\pi}{2}) + c$$

Solving for a, b and c:

$$a = \frac{1}{\pi^2}, b = -\frac{1}{\pi}, c = \frac{5}{4}$$

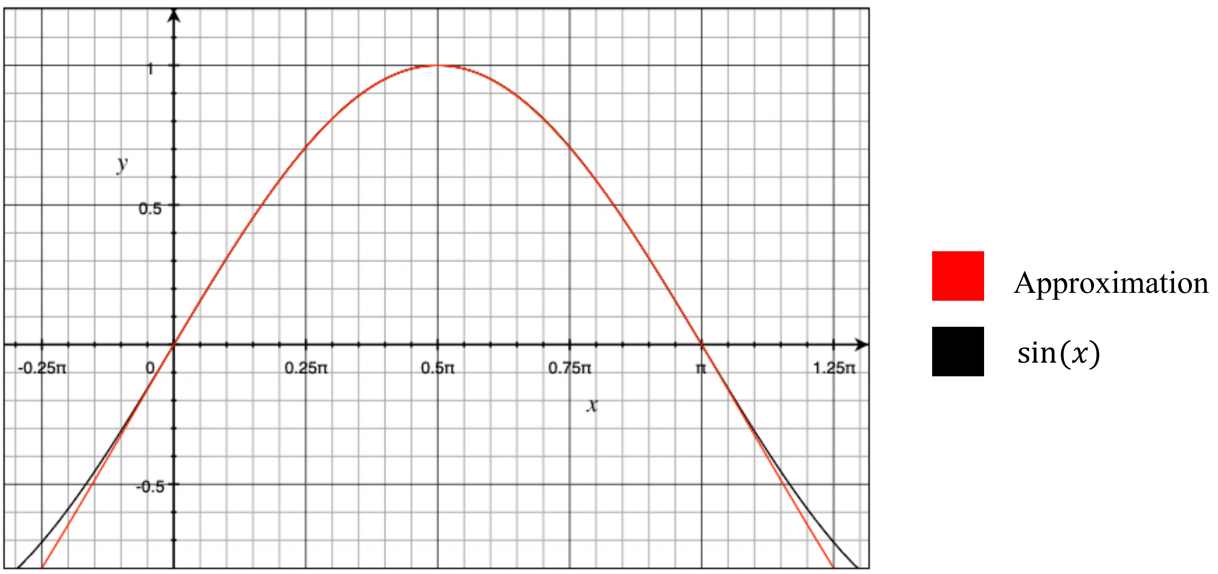
Plugging the values into the scaling function, we get:

$$\frac{\frac{4x(\pi - x)}{\pi^2}}{\frac{1}{\pi^2}x^2 - \frac{1}{\pi}x + \frac{5}{4}}$$

This formula simplifies to:

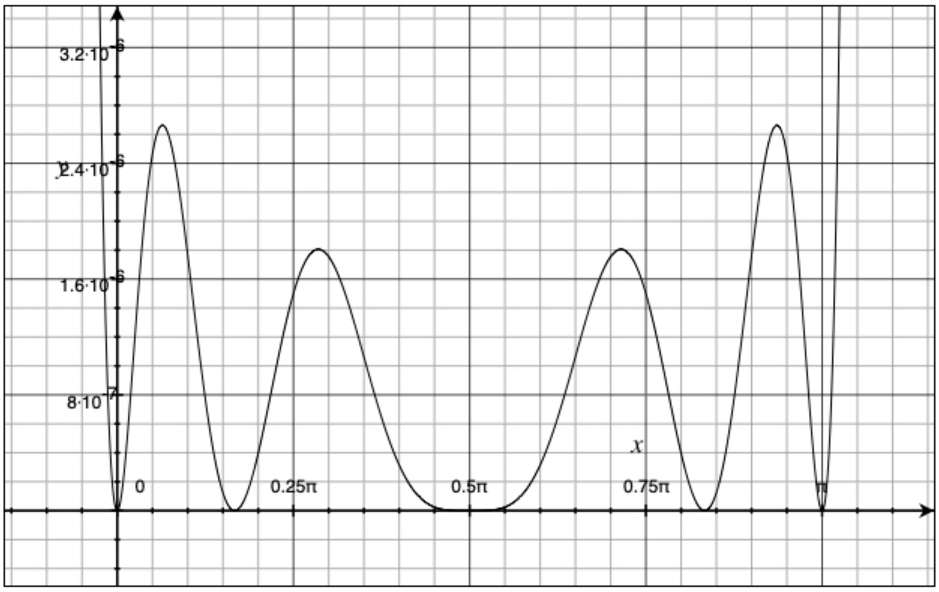
$$sin(x) \approx \frac{16x(\pi-x)}{4x^2-4\pi x+5\pi^2} = \frac{16x(\pi-x)}{5\pi^2-4x(\pi-x)}$$

Plotting Bhaskara's approximation over the true sine function gives us two graphs that are nearly indistinguishable in their shape between 0 and π (Graph 7). The error function has a maximum value of 2.663×10^{-6} (Graph 8), which means that it will be accurate up to nearly 3 decimal places.



Graph 7: Bhaskara I's approximation plotted against sin(x)

The limitations of Bhaskara's approximation are similar to the ones faced by the simple quadratic method — a parabolic approximation cannot represent a wide range of values for $sin(x)$, it remains limited to a maximum range of π, after which its inaccuracy



Graph 8: Squared-error in Bhaskara I's approximation

increases at a very large rate. As such, Bhaskara's approximation would provide us with a higher level of accuracy than our simple quadratic approximation or a comparable Taylor polynomial in the limited range of 0 to π, but it would not be appropriate in other cases.

Conclusion

I began this investigation with a certain curiosity about the nature of transcendental functions and exploring the various methods in which we can approximate trigonometric expressions in algebraic terms helped me understand how such functions fundamentally differ from their simpler non-transcendental counterparts. While we may take the ability to calculate trigonometric values for granted due to widespread access to digital calculators, attempting to evaluate them manually using algebraic approximations can often hinder the accuracy of our calculation. We can improve the accuracy and range of such approximations by either using higher degree approximations or scaling our approximations closer to the values of the real function, but there is a constant trade-off — making our approximations more complicated makes them inherently less useful as it becomes harder to calculate them manually.

The infinitely repetitive nature of the trigonometric $sin(x)$ was one that turned out to be impossible to replicate in terms of finite algebraic terms, but the Taylor series provides us with a versatile solution that can be expanded to any required range. The fact that Bhaskara I's method could evaluate a trigonometric function to a high level of accuracy centuries before the invention of the first calculators showcases the effectiveness of analogue mathematical methods despite the algebraically "inexpressible" nature of transcendental functions. Ultimately, these approximations bring to light the advantages and limitations of algebraic functions — their ability to both be directly calculated and to approximate the shapes of other relations and patterns is a powerful mathematical tool that allows us to better understand and utilise even the most elusive mathematical concepts, but they are inhibited by their finite nature and the need to be explicitly defined.

These limitations suggest further scope for exploration in developing trigonometric approximations — we could use computer programming or equations in terms of mod functions to generalise such approximations to a much larger range of values for x. We can also derive approximations for other trigonometric functions such as $\cos(x)$ and $\tan(x)$ using trigonometric identities. Other lines of exploration may include comparing these elementary methods to advanced calculation algorithms such as CORDIC[6], which is commonly used in graphing calculators to compute trigonometric functions.

[6] Sultan, Alan. "CORDIC: How Hand Calculators Calculate." *The College Mathematics Journal*, vol. 40, no. 2, 2017, pp. 87-92, doi:10.1080/07468342.2009.11922342. Accessed 12 Feb. 2020.

Works Cited

Electronic Sources

Berry, Nick. "Approximating the Sine Function." *Data Genetics*, 1 July 2019, datagenetics.com/blog/july12019/index.html. Accessed 3 Dec. 2019.

O' Connor, J. J., and E. F. Robertson. "Bhaskara I." *Bhaskara I (about 600 - about 680)*, mathshistory.st-andrews.ac.uk/Biographies/Bhaskara_I.html. Accessed 3 Dec. 2019.

Sultan, Alan. "CORDIC: How Hand Calculators Calculate." *The College Mathematics Journal*, vol. 40, no. 2, 2017, pp. 87-92, doi:10.1080/07468342.2009.11922342. Accessed 12 Feb. 2020.

"Taylor Series Approximation." *Brilliant Math & Science Wiki*, brilliant.org/wiki/taylor-series-approximation/. Accessed 12 Feb. 2020.

"Transcendental Function." *Encyclopædia Britannica*, Encyclopædia Britannica, Inc., 15 Apr. 2011, www.britannica.com/science/transcendental-function. Accessed 27 Jan. 2020.